FREQUENCY SHIFT OF RECEIVED WAVES

V. I. Krylovich

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Phenomena are examined which cause the frequency of a wave process to change from source to receiver. Analytical expressions are derived for the magnitude of the frequency shift and special cases are considered.

A change in the frequency of the signal at the receiver from that at the source, regardless of the nature of the latter, can be due to various causes. This is generally a very complex phenomenon with many anomalies which are usually disregarded in practice.

The appearance of a frequency shift is always related to a variation of system parameters in time, the system consisting of the input signal, the source, the medium, and the receiver. The system parameters include the frequency of the input signal, the relative position of source and receiver, the velocity of wave propagation through the medium, the velocity of the medium, etc.

Best known and most widely utilized is the Doppler effect [1], due to the relative motion of source and receiver. This effect is of a purely kinematic origin and is related to the choice of the reference system. It has been analyzed theoretically in several monographs [2, 3]. There is also a Doppler effect when an object reflects incident waves while moving relative to the source and the receiver, which has found application in radar and sonar.

There is furthermore a frequency shift due to a variation, in time, of the properties of the medium through which the wave propagates. The first one to observe this effect was Michelson [4], but the analytical expressions he has derived for this frequency shift are approximate. In other studies [5, 6] on the subject this effect has been called the nonstationary Doppler effect.

A frequency shift occurs also when, under otherwise constant conditions, the frequency of waves radiated into the medium varies in time. The effect in this case is due to a time delay of the received signal. When there is dispersion, there occurs an additional frequency shift due to this phenomenon. Finally, a frequency occurs during acceleration of the medium between source and receiver of waves. This effect could be called the convective Doppler effect.

A unified analysis of all these phenomena was attempted before [7]. However, as the author of that study himself notes, the fundamental equation there is approximate and this limits its range of applicability so as to detract from the generality of the final conclusions. Let us therefore examine the problem more rigorously. In the idealization for this purpose, we will disregard dispersion as well as reverberation interference and will consider the one-dimensional case: a plane wave propagating along the X axis through a homogeneous medium at a velocity which is a function of time $c = c(\tau)$. The analysis will be performed in two stages: first for a stationary medium and then of the convective effect separately. For simplification, furthermore, we will consider elastic waves. A transition to electromagnetic waves with consideration of the theory of relativity does not present any particular difficulty.

<u>Frequency Shift in a Stationary Medium</u>. We consider the segment of the medium between source and receiver where the velocity of wave propagation $c = c(\tau)$ is a continuous integrable function of time. The source and the receiver move along the X axis according to the laws $X_1(\tau)$ and $X_2(\tau)$, respectively. The source emits plane elastic waves at the frequency $f_1(\tau)$, the frequency of waves arriving at the receiver will be denoted as $f_2(\tau)$.

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Let a wave with a phase φ be emitted by the source at time τ_1 and arrive at the receiver at time τ_2. There is the obvious relation (provided that the velocity of the wave is higher than the velocity of the receiver)

$$X_2(\tau_2) - X_1(\tau_1) = \int_{\tau_1}^{\tau_2} c(\tau)\, d\tau. \tag{1}$$

This equation takes into account a change in the velocity of the wave during propagation of the latter through a monitored medium.

Let later a wave with a phase $\varphi + \Delta\varphi$ be emitted at time $\tau_1 + \Delta\tau_1$ and received at time $\tau_2 + \Delta\tau_2$; then analogously

$$X_2(\tau_2 + \Delta\tau_2) - X_1(\tau_1 + \Delta\tau_1) = \int_{\tau_1+\Delta\tau_1}^{\tau_2+\Delta\tau_2} c(\tau)\, d\tau. \tag{2}$$

Subtracting the left-hand side and the right-hand side of Eq. (1) from those of Eq. (2), respectively, we obtain

$$[X_2(\tau_2 + \Delta\tau_2) - X_2(\tau_2)] - [X_1(\tau_1 + \Delta\tau_1) - X_1(\tau_1)] = \int_{\tau_2}^{\tau_2+\Delta\tau_2} c(\tau)\, d\tau - \int_{\tau_1}^{\tau_1+\Delta\tau_1} c(\tau)\, d\tau. \tag{3}$$

We are interested in the dependence of $\Delta\tau_2$ on $\Delta\tau_1$, which characterizes the relation between the periods or the frequencies of the emitted and received (the same) wave. Accordingly, using the theorem of the mean, we write

$$\Delta\varphi \equiv 2\pi f_1(\xi)\, \Delta\tau_1 = 2\pi f_2(\eta)\, \Delta\tau_2, \tag{4}$$

where $\tau_1 \leqslant \xi \leqslant \tau_1 + \Delta\tau_1$, $\tau_2 \leqslant \eta \leqslant \tau_2 + \Delta\tau_2$. Converging to the limit as $\Delta\varphi \to 0$ and considering that

$$\lim_{\Delta\varphi\to 0} \Delta\tau_1 = 0, \qquad \lim_{\Delta\tau_1\to 0} \Delta\tau_2 = 0,$$

we obtain

$$\lim_{\Delta\tau_1\to 0} \frac{\Delta\tau_2}{\Delta\tau_1} = \lim_{\Delta\tau_1\to 0} \frac{f_1(\xi)}{f_2(\eta)} = \frac{f_1(\tau_1)}{f_2(\tau_2)}. \tag{5}$$

On the other hand, we can also obtain the same limit from relation (3). For this we use the Taylor series expansion

$$X_2(\tau_2 + \Delta\tau_2) = X_2(\tau_2) + \frac{dX_2(\tau_2)}{d\tau} \Delta\tau_2 + \ldots, \tag{6}$$

$$X_1(\tau_1 + \Delta\tau_1) = X_1(\tau_1) + \frac{dX_1(\tau_1)}{d\tau} \Delta\tau_1 + \ldots \tag{7}$$

and the theorem of the mean, according to which

$$\int_{\tau_2}^{\tau_2+\Delta\tau_2} c(\tau)\, d\tau = c(\alpha)\, \Delta\tau_2, \quad \tau_2 \leqslant \alpha \leqslant \tau_2 + \Delta\tau_2, \tag{8}$$

$$\int_{\tau_1}^{\tau_1+\Delta\tau_1} c(\tau)\, d(\tau) = c(\beta)\, \Delta\tau_1, \quad \tau_1 \leqslant \beta \leqslant \tau_1 + \Delta\tau_1. \tag{9}$$

The result is

$$\lim_{\Delta\tau_1\to 0} \frac{\Delta\tau_2}{\Delta\tau_1} = \frac{c(\tau_1) - u_1(\tau_1)}{c(\tau_2) - u_2(\tau_2)}, \tag{10}$$

where $u_1(\tau_1) = \partial X_1(\tau_1)/\partial\tau$ is the velocity of the source at the instant it emits a certain wave and $u_2(\tau_2) = dX_2(\tau_2)/d\tau$ is the velocity of the receiver at the instant this wave arrives there. A velocity is regarded as positive in the direction of the X axis. Comparing relations (10) and (5), we obtain the final expression

$$f_2(\tau_2) = f_1(\tau_1) \frac{c(\tau_2) - u_2(\tau_2)}{c(\tau_1) - u_1(\tau_1)} \tag{11}$$

for the frequency of received waves. This expression is exact in terms of the idealization of the problem. It yields the frequency of a received signal (wave) as a function of the frequency at which that signal was emitted.

Let us consider a few special cases.

1. Let $f_1 = f_0 = \text{const}$. Then relation (11) becomes

$$f_2(\tau_2) = f_0 \frac{c - u_2(\tau_2)}{c - u_1(\tau_1)}. \tag{12}$$

Expression (12) describes the conventional Doppler effect, at constant u_1 and u_2 becoming the classical expression found in textbooks. At the same time, it is a more general expression including also the case of nonuniform motion of the source and the receiver. The frequency shift at the receiver $\Delta f = f_2(\tau) - f_0$ at any instant of time is

$$\Delta f(\tau) = f_0 \frac{u_1(\tau - \tau_3) - u_2(\tau)}{c - u_1(\tau - \tau_3)}, \tag{13}$$

where τ_3 is the time taken by the wave received at a given instant to travel from the source (delay time).

2. Let $f_1 = f_0 = \text{const}$ and $u_1 = u_2 = 0$ (stationary radiator and receiver). Then

$$f_2(\tau_2) = f_0 \frac{c(\tau_2)}{c(\tau_1)}, \tag{14}$$

which means the frequency change at the receiver is caused by a variation of the wave propagation velocity in time. This is, indeed, the nonstationary Doppler effect [5, 6].

3. Let $c = \text{const}$, $u_1 = u_2 = 0$, and $f_1 = f_1(\tau)$. Then expression (11) yields

$$f_2(\tau_2) = f_1(\tau_1). \tag{15}$$

This equality means the frequency does not change due to propagation of the wave through the monitored segment of the medium. However, a comparison of frequencies f_2 and f_1 at one and the same instant of time will reveal that they differ. Exactly, if the emission frequency is $f(\tau)$, then the reception frequency at that same instant of time will be $f(\tau - \tau_3)$ and the frequency shift will be

$$\Delta f(\tau) = f(\tau - \tau_3) - f(\tau). \tag{16}$$

When $\Delta f \tau_3 << 1$, then we can write

$$\Delta f(\tau) \approx -\frac{df}{d\tau}\tau_3 = -\frac{df}{d\tau}\frac{L}{c}, \tag{17}$$

where $L = X_2 - X_1$.

When the medium (delay time) is dispersive, i.e., $c = c(f)$, there appears an additional frequency shift. Calculations for $L = \text{const}$ yield in this case a frequency shift

$$\Delta f(\tau) = \frac{f(\tau - \tau_3)}{1 - \left(\frac{dc}{df}\frac{df}{d\tau}\frac{L}{c^2}\right)_{\tau-\tau_3}} - f(\tau) \approx -\frac{1 - f(\tau)\frac{dc}{df}\frac{1}{c}}{1 - \frac{dc}{df}\frac{df}{d\tau}\frac{L}{c^2}}\frac{df}{d\tau}\frac{L}{c} \tag{18}$$

at any instant of time τ. For real conditions the second term in the denominator here is an order of magnitude smaller than unity so that the additional frequency shift due to dispersion $\Delta f_d(\tau)$ becomes

$$\Delta f_d(\tau) = f(\tau)\frac{L}{c^2}\frac{dc}{df}\frac{df}{d\tau}. \tag{19}$$

Convective Frequency Shift. At a point A (Fig. 1) let there be located a source of signals propagating as waves through the medium from point A to point B, the length of the segment AB being $L = \text{const}$. When the medium between points A and B moves, then the waves will be drifting with the medium and at point B will arrive not a wave emitted in the direction AB but one emitted in some other direction AC as shown in Fig. 1. Let us assume that a continuous wave is emitted at a constant frequency f_0. The phase difference between received and emitted oscillations is obviously

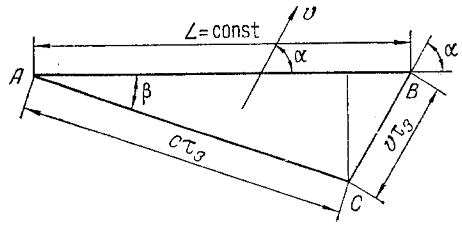

Fig. 1. Calculation of the travel time from source to receiver.

$$\varphi_B - \varphi_A = -2\pi f_0 \tau_3. \tag{20}$$

According to the diagram in Fig. 1,

$$L = c\tau_3 \cos\beta + v\tau_3 \cos\alpha, \tag{21}$$

where v is the magnitude of the velocity of the moving medium and α is the angle which the vector of this velocity makes with the direction AB. From this relation, taking into account the equality

$$c\tau_3 \sin\beta = v\tau_3 \sin\alpha, \tag{22}$$

we obtain

$$\tau_3 = \frac{L}{\sqrt{c^2 - v^2\sin^2\alpha} + v\cos\alpha}. \tag{23}$$

We will assume that the velocity of the stream (medium) varies in time. Then during a time dτ the magnitude of the time delay will change by $d\tau_3$ and the phase difference will increase additionally by

$$d\varphi = -2\pi f_0 d\tau_3 \tag{24}$$

with a frequency shift occuring which can be called a convective Doppler effect. The magnitude of this frequency shift is

$$\Delta f = \frac{1}{2\pi}\frac{d\varphi}{d\tau}. \tag{25}$$

Differentiating expression (23) and taking into account relations (24) and (25), we obtain expressions for dφ and Δf which in the final form become

$$d\varphi = 2\pi f_0 L \left[\frac{\cos\alpha\sqrt{c^2 - v^2\sin^2\alpha} - v\sin^2\alpha}{(\sqrt{c^2 - v^2\sin^2\alpha} + v\cos\alpha)^2\sqrt{c^2 - v^2\sin^2\alpha}}\,dv - \frac{v\sin\alpha}{(\sqrt{c^2 - v^2\sin^2\alpha} + v\cos\alpha)\sqrt{c^2 - v^2\sin^2\alpha}}\,d\alpha \right], \tag{26}$$

$$\Delta f = f_0 L \left[\frac{\cos\alpha\sqrt{c^2 - v^2\sin^2\alpha} - v\sin^2\alpha}{(\sqrt{c^2 - v^2\sin^2\alpha} + v\cos\alpha)^2\sqrt{c^2 - v^2\sin^2\alpha}}\,\frac{dv}{d\tau} - \frac{v\sin\alpha}{(\sqrt{c^2 - v^2\sin^2\alpha} + v\cos\alpha)\sqrt{c^2 - v^2\sin^2\alpha}}\,\frac{d\alpha}{d\tau} \right]. \tag{27}$$

It is evident that an additional phase lead as well as frequency shift can occur due to a variation of the stream velocity as well as due to a change in the stream direction. Let us consider some special cases. When the stream velocity does not change direction, then $d\alpha/d\tau = 0$ and expression (27) simplifies. With α = 0 or α = π, we have for the longitudinal effect respectively

$$\Delta f_{\parallel} = \frac{f_0 L}{(c+v)^2}\frac{dv}{d\tau}, \quad \alpha = 0, \tag{28}$$

$$\Delta f_{\parallel} = -\frac{f_0 L}{(c-v)^2}\frac{dv}{d\tau}, \quad \alpha = \pi. \tag{29}$$

With α = π/2 or α = 3π/2, we have for the transverse effect

$$\Delta f_{\perp} = -\frac{f_0 L v}{(c^2 - v^2)^{3/2}}\frac{dv}{d\tau}. \tag{30}$$

It is easy to see that $\Delta f_{\perp}/\Delta f_{\parallel} \approx v/c$ when $v \ll c$, i.e., the transverse effect is then much smaller than the longitudinal effect.

The phase difference between received and emitted waves is generally calculated by integration of expression (26). For the special cases of longitudinal and tranverse flow of the medium we easily obtain from expressions (28)-(30)

$$\Delta\varphi_{\parallel} = \pm 2\pi f_0 L \frac{\Delta v}{(c \pm v_0)(c \pm v_0 \pm \Delta v)}, \tag{31}$$

$$\Delta\varphi_{\perp} = 2\pi f_0 L \frac{\sqrt{c^2-(v_0+\Delta v)^2}-\sqrt{c^2-v_0^2}}{\sqrt{c^2-v_0^2}\sqrt{c^2-(v_0+\Delta v)^2}}, \tag{32}$$

where v_0 is the magnitude of the stream velocity at the initial instant of time (beginning of measurement), and the signs "+," "−" in expression (31) refer to $\alpha = 0$, $\alpha = \pi$, respectively.

A few comments on the practical application of these effects are in order. The nonstationary effect is already used for the study of convective heat transfer and some other phenomena [8, 9]. The frequency shift describable by expressions (16) and (17) can be utilized in an analysis of electric signals with frequency-phase modulation, for determining the time instability of oscillators. With the convective effect one can study transient flow and other processes.

NOTATION

c, velocity of wave propagation through the medium; f, frequency; X, a space coordinate; τ, time, φ, phase; u_1 and u_2, velocity of the source and of the receiver, respectively; v, velocity of the medium; and τ_3, wave travel time from source to receiver.

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